Quantum Scattering in Two Black Hole Moduli Space

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Abstract

We discuss the quantum scattering process in the moduli space consisting of two maximally charged dilaton black holes. The black hole moduli space geometry has different structures for arbitrary dimensions and various values of dilaton coupling. We study the quantum effects of the different moduli space geometries with scattering process. Then, it is found that there is a resonance state on certain moduli spaces.

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I. INTRODUCTION

Recently the study of black hole moduli space has attracted much attention. Quantum black holes have been studied by means of quantum fields or strings interacting with a single black hole. In the past several years the quantum mechanics of an arbitrary number n of supersymmetric black holes has been focused upon. Configurations of n static black holes parametrize a moduli space. The low-lying quantum states of the system are governed by quantum mechanics in the moduli space. The effective theories in moduli space of Reissner-Nordström multi-black holes were constructed in [1]. Recently, (super)conformal quantum mechanics are constructed in the moduli space of four- and five-dimensional multi-black holes in [2], [3]. Further, we found that the quantum mechanics in moduli space of multi-black holes with dilaton coupling in diverse dimensions have the conformal symmetry [4].

The scattering in moduli space was discussed by Ferrell and Eardley [5], where the black hole-black hole scattering and coalescence in the moduli space, which is constructed from (3+1) dimensional Reissner-Nordstöm black holes, was calculated. The black hole moduli spaces geometry with dilaton coupling in $(N + 1)$ dimensions was discussed by one of the present authors [6]. In these works the structure of the moduli space geometry is found to be different for dimensions and values of dilaton coupling.

In this paper, we consider the quantum effects from the various moduli structures with the scattering in moduli space. In section 2, we discuss the moduli space metric of two black hole system with dilaton coupling in $(3 + 1)$ dimensions. In section 3, we investigate the quantum mechanics in the moduli space. From the feature of potential, we discuss whether an incoming particle is scattered away, or coalesces. In section 4, we consider the scattered-away processes. Using the WKB approximation, we calculate the phase shift and the deflection angle. Then we find the quantum effect in the moduli space. In the special cases of the moduli space, there are resonance states, which are discussed in section 5. In section 6, we give discussion and conclusion.

II. THE MODULI SPACE METRIC FOR THE SYSTEM OF MAXIMALLY CHARGED DILATON BLACK HOLES

In this section, we consider the moduli space of the Einstein-Maxwell-dilaton system along the technique discussed by Ferrell and Eardley [5]. The Einstein-Maxwell-dilaton system contains a dilaton field ϕ coupled to a $U(1)$ gauge field A_μ besides the Einstein-Hilbert gravity. In the $(3+1)$ dimensions, the action for the fields with particle sources is

$$S = \int d^4x \frac{\sqrt{-g}}{16\pi} \left[R - 2(\nabla\phi)^2 - e^{-2a\phi}F^2 \right] - \sum_{A=1}^{n} \int ds_A \left(m_A e^{-2a\phi} + Q_A \mathbf{A} \frac{d\mathbf{x}_A}{ds_A} \right), \qquad (1)$$

where R is the scalar curvature and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. ds_A is the line element of the center of the A-th black hole. m_A and Q_A are the mass and the electric charge of the A-th black hole. We set the Newton constant $G = 1$. The dilaton coupling constant a can be assumed to be a positive value.

The metric for the n-body system of maximally-charged dilaton black holes has been known as [6] [1]

$$ds^2 = -U^{-2}(\mathbf{x})dt^2 + U^2(\mathbf{x})d\mathbf{x}^2, \qquad (2)$$

where

$$U(\mathbf{x}) = (F(\mathbf{x}))^{1/(1+a^2)}, \qquad (3)$$

$$F(\mathbf{x}) = 1 + \sum_{A=1}^{n} \frac{\mu_A}{|\mathbf{x} - \mathbf{x}_A|}. \qquad (4)$$

Using these expressions, the vector one form and dilaton configuration are written as

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[1] In the special case of $a = 0$ for $N = 3$, the solution becomes the Papapetrou-Majumdar solution [7], [8]. Generally, these solutions have singularities. We, however, consider a kind of soliton with three long-range forces balanced (the gravitational force, the electric force and the dilatonic force) as the 'black hole'.

$$A = \sqrt{\frac{1}{1 + a^2}} \left(1 - \frac{1}{F(\mathbf{x})} \right) dt, \tag{5}$$

$$e^{-2a\phi} = (F(\mathbf{x}))^{2a^2/(1+a^2)}. \tag{6}$$

In this solution, the asymptotic value of ϕ is fixed to be zero.

The electric charge Q_A of each black hole are associated with the corresponding mass m_A by

$$m_A = \frac{1}{1 + a^2} \mu_A, \tag{7}$$

$$|Q_A| = \sqrt{\frac{1}{1 + a^2}} \mu_A. \tag{8}$$

We consider that the perturbed metric and potential can be written in the form

$$ds^2 = -U^{-2}(\mathbf{x})dt^2 + 2\mathbf{N}d\mathbf{x}dt + U(\mathbf{x})d\mathbf{x}^2, \tag{9}$$

$$A = \sqrt{\frac{1}{1 + a^2}}(1 - \frac{1}{F(\mathbf{x})})dt + \mathbf{A}d\mathbf{x}, \tag{10}$$

where $U(\mathbf{x})$ and $F(\mathbf{x})$ are defined by (3) and (4). We have only to solve linearized equations with perturbed sources up to $O(v)$ for N_i and A_i. (Here v represents the velocity of the black hole as a point source.) We should note that each source plays the role of a maximally charged dilaton black hole.

Solving the Einstein-Maxwell equations and substituting the solutions, the perturbed dilaton field and sources to the action (1) with proper boundary terms, we get the effective action up to $O(v^2)$ for n-maximally charged dilaton black hole system

$$S = \int dt \left(\frac{1}{2} \sum_A^n m_A \mathbf{v}_A^2 \right) + \frac{3 - a^2}{8\pi(1 + a^2)^2} \int d^4x (F(\mathbf{x}))^{\frac{2(1-a^2)}{1+a^2}}$$

$$\times \sum_{A,B=1}^n \frac{\mu_A \mu_B}{|\mathbf{r}_A|^3 |\mathbf{r}_B|^3} \left[\frac{1}{2}(\mathbf{r}_A \cdot \mathbf{r}_B)|\mathbf{v}_A - \mathbf{v}_B|^2 - (\mathbf{r}_A \times \mathbf{r}_B) \cdot (\mathbf{v}_A \times \mathbf{v}_B) \right], \tag{11}$$

where $\mathbf{r}_A = \mathbf{x} - \mathbf{x}_A$. $F(\mathbf{x})$ is defined by (4).

In general, a naive integration in equation (11) diverges. Therefore, we regularize the divergent terms proportional to $\int d^3x \delta^3(x)/|x|^p$ $(p > 0)$ which appear when the integrand is expanded must be regularized [9]. We set them to zero.

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After regularization, the effective action for two body system (consisting of black holes labeled with a and b) can be rewritten as

$$S_{2B} = \int dt \frac{1}{2} \mu \mathbf{v}^2 \left[1 - \frac{M}{\mu} - \frac{(3-a^2)M}{r} + \frac{M}{m_a} \left(1 + (1+a^2) \frac{m_a}{r} \right)^{(3-a^2)/(1+a^2)} \right.$$
$$\left. + \frac{M}{m_b} \left(1 + (1+a^2) \frac{m_b}{r} \right)^{(3-a^2)/(1+a^2)} \right], \qquad (12)$$

where $M = m_a + m_b$, $\mu = m_a m_b / M$, $\mathbf{v} = \mathbf{v}_a - \mathbf{v}_b$ and $r = |\mathbf{x}_a - \mathbf{x}_b|$. Thus the metric of the three dimensional moduli space for two-body system is

$$g_{ij} = \gamma(r) \delta_{ij}, \qquad (13)$$

with

$$\gamma(r) = 1 - \frac{M}{\mu} - \frac{(3-a^2)M}{r} + \frac{M}{m_a} \left(1 + \frac{(1+a^2)m_a}{r} \right)^{\frac{3-a^2}{1+a^2}} + \frac{M}{m_b} \left(1 + \frac{(1+a^2)m_b}{r} \right)^{\frac{3-a^2}{1+a^2}}.$$
$$(14)$$

The moduli space metric (13) depends on the dilaton coupling. We notice that the behavior of the moduli metric changes at $a^2 = 1/3$, where the moduli space has a critical structure. Therefore, the moduli space geometry has the various structure [6].

III. QUANTUM MECHANICS IN TWO-BLACK HOLE MODULI SPACE

We consider the quantum scattering in this moduli space. The effective theories of quantum mechanics in moduli space of Reissner-Nordstöm black holes were constructed in [10].

Let us introduce a wave function Ψ in the moduli space, which obeys the Schrödinger equation

$$i\hbar \frac{d\Psi}{dt} = -\frac{\hbar^2}{2\mu} \nabla^2 \Psi, \qquad (15)$$

where ∇^2 is the covariant Laplacian constructed from the moduli space metric (13).

The partial wave in a stationary state is

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$$\Psi = \psi_{ql}(r)Y_{lm}(\theta, \phi)\exp(-iEt/\hbar), \tag{16}$$

where $Y_{lm}(\theta, \phi)$ is the spherical harmonic function. We define the energy and variables as

$$E = \frac{\hbar^2 q^2}{2\mu}, \tag{17}$$

$$R = \int \sqrt{\gamma}dr, \tag{18}$$

$$\psi = \frac{\chi}{r\sqrt{\gamma}}. \tag{19}$$

The Schrödinger equation (16) is rewritten as

$$\frac{d^2\chi}{dR^2} + (q^2 - V)\chi = 0, \tag{20}$$

where the potential V is represented as

$$V = \frac{\gamma(r\gamma')' - r(\gamma')^2}{2r\gamma^3} + \frac{l(l+1)}{r^2\gamma}. \tag{21}$$

Here $'$ stands for $\frac{d}{dr}$ and l is the angular momentum.

To study the scattering process, we find the general view of the potential in the moduli space. The potentials for dilaton couplings $a^2 = 0, 1/3, 1/2$ and 1 are plotted in Fig. 1 and Fig. 2.

From Fig. 1, we can find that there is the potential barrier in the case of $a^2 = 0$ for all values of l and $a^2 = 1/3, 1/2$ for $l = 0$. If the potential barrier exists, the incoming particle approaching to the origin $r = 0$ of the moduli space is scattered or coalesces. On the other hand, in the case of $a^2 = 1/3, 1/2$ for all values of $l \neq 0$ and $a^2 = 1$ with any l, the incoming particle is always scattered away.

Generally, if the angular momentum is zero ($l = 0$) the potential has the barrier in the case of $a^2 < 1$. For $a^2 < 1/3$, the variable R diverges to minus infinity at the limit of $r \rightarrow 0$. Then the particle going over the potential barrier can reach the minus infinity, $R \rightarrow -\infty$, so that, the incoming particle coalesces with the target black hole. In the case of $a^2 > 1/3$, the variable R has the finite value at the limit of $r \rightarrow 0$. The coordinate R taking the infinite

value means that the moduli space structure has the throat of infinite length [6]. [2] In the case of $1/3 < a^2 < 1$, the incoming particle through the potential barrier can reach the point of $r = 0$ for the finite time and cannot stay at $r = 0$. Therefore, the incoming particle is reflected at $r = 0$ and scattered away out of the barrier, so that, does not coalesce. For the scattering of the incoming particle, we can guess that the potential becomes infinity at the point of $r = 0$. The example of such a potential is shown in Fig. 2 C. We notice that the value of $a^2 = 1/3$ is the critical point of moduli structures [6].

IV. SCATTERING IN TWO-BLACK HOLE MODULI SPACE

First, for simplicity, we will study the scattering-away processes in the case of $a^2 = 1/3$ and $a^2 = 1$ for arbitrary values of angular momenta. As the first study of quantum effects, we discuss the scattering process in the WKB approximation. In the WKB approximation, the phase shift in the scattering process is given by

$$\delta_l = -qR_0 + \int_{R_0}^{\infty} dR \left(\sqrt{q^2 - V(R) - \frac{1}{4R^2}} - q \right) + \frac{2l+1}{4}\pi, \tag{22}$$

where R_0 is the solution of $V(R_0) = q^2 - \frac{1}{4R_0^2}$. The partial cross section is

$$\sigma_l = \frac{4\pi}{q^2}(2l+1)\sin^2 \delta_l. \tag{23}$$

The deflection angle is

$$\Theta = \pi + \int_{R_0}^{\infty} dR \frac{\partial}{\partial l} \sqrt{q^2 - V(R) - \frac{1}{4R^2}}. \tag{24}$$

These equations (22)-(24) are evaluated with respect to the outside of the potential barrier, where we ignore the effect of the inside of potential barrier. Thus, these are meaningful in the case of large q when the potential barrier does not influence the calculation strongly. The low velocity and large q can be realized if $\hbar \ll 1$, because $v \sim \hbar q/\mu$.

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[2]Exactly, the radial coordinate describing the moduli space structure in [6] is different from R. However, these behaviors at $r \to 0$ are similar.

The phase shifts δ_l and the deflection angles Θ for $q = 0.1$ and 0.2 are plotted in Fig. 3. The solid line in Fig. 3 A2, B2 represents the classical deflection angle which is obtained from the impact parameter [11]. For $a^2 = 1$, the effects for the different value of q are not found, and the deflection angle almost corresponds to the classical deflection angle. Although, for $a^2 = 1/3$, we can find that the behaviors of phase shift and deflection angle depend on the incoming particle energy. Then the deflection angle is different from the classical deflection angle at the small value of l/q. These effects are interpreted as the quantum effects arisen from the difference of the moduli space structures in the scattering process. Although we used the WKB semi-classical approximation in this analysis, the quantum effects can be obtained sufficiently.

V. RESONANCE

As seen in section III, the moduli space for $a^2 > 1/3$ has the finite region toward $r = 0$, because the R has a finite value in the limit $r \to 0$ under the integration. The incoming particle through the potential barrier cannot stay at the point of $r = 0$. Therefore, the coming particle is reflected at the point of $r = 0$ and scattered away. We can guess that the potential becomes infinity at the point of $r = 0$. Further, the potential for $a^2 < 1$ has the finite barrier. The example of the potential is shown in Fig. 2 C. The typical potential form for $1/3 < a^2 < 1$ is shown in Fig. 4. The energy of an incoming particle is presented by E in Fig. 4. At the point of $R = R_3$, the original variable r equal to zero. For $1/3 < a^2 < 1$, the potential has the barrier and three turning points. In this scattering process on the moduli space for $1/3 < a^2 < 1$, we can expect that there is the resonance state.

In this section, we discuss the resonance state in the scattering process. The potential has three classical turning points. In Fig. 4, there is a classically allowed region II between R_3 and R_2 and for $R > R_1$ and a classically inaccessible region I between R_2 and R_1. In the WKB approximation, the scattering amplitude is given [12]

$$S_l = \exp(2i\delta_1)\frac{1 + \bar{N}(i\beta)\exp(2iW_{32})}{N(i\beta) + \exp(2iW_{32})},$$ (25)

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where δ_1 is the WKB phase shift (22) evaluated with respect to the outermost turning point R_1, and

$$N(i\beta) = \frac{\sqrt{2\pi}\exp(-\pi\beta/2 + i(\beta\ln\beta - \beta))}{\Gamma(\frac{1}{2} + i\beta)}, \tag{26}$$

$$\bar{N} = N^*, \tag{27}$$

with

$$\beta = \frac{i}{\pi}\int_{R_1}^{R_2}\sqrt{\chi(R)}dR = \frac{1}{\pi}\int_{R_1}^{R_2}dR\sqrt{-(q^2 - V(R) - \frac{1}{4R^2})}, \tag{28}$$

$$W_{32} = \int_{R_3}^{R_2}dR\sqrt{q^2 - V(R) - \frac{1}{4R^2}}. \tag{29}$$

The partial cross section is represented by

$$\sigma_l = 4\pi(2l + 1)|f_l|^2, \tag{30}$$

where

$$f_l = \frac{1}{2iq}(S_l - 1). \tag{31}$$

In the cases of $a^2 = 2/5$, $a^2 = 9/20$ and $a^2 = 1/2$, the partial cross sections σ_0 are shown in Fig. 5.

Clearly, there is the resonance state. The quasi-bound state exists inside the potential barrier: The quasi-bound states for $a^2 = 2/5$ and $a^2 = 9/20$ exist below the maximal value of the potential barrier, $V \approx 0.0619$ and $V \approx 0.0641$ respectively. The quasi-bound state for $a^2 = 1/2$ exists near the maximal value $V \approx 0.0665$.

Also, the width of resonance for $a^2 = 1/2$ is larger than for $a^2 = 9/20$ and $a^2 = 2/5$. Here the width of the resonance Γ_0 is given by

$$\Gamma_0 = \frac{P}{T}, \tag{32}$$

where P is the barrier penetration factor and T is the period of the classical motion in the potential pocket inside the barrier:

$$P = (1 + \exp 2\pi\beta)^{-1}, \tag{33}$$

$$T = \left(\frac{\partial W_{32}}{\partial E}\right)_{E_r}, \tag{34}$$

where E_r is the resonance energy. The width of the resonance is related to the time of classical motion of the test particle. From the width of resonance, we can guess that the time of the classical motion in moduli space for $a^2 = 2/5$ is longer than the case of $a^2 = 1/2$. Fig. 6 shows the surface of moduli space for $a^2 = 2/5$ and $1/2$ in the (3+1) dimensions. The moduli space geometry becomes deep and sharp for small values of a^2. Then, we suppose that the time of classical motion in moduli space for small a^2 is also long, because the trajectories of the classical motion becomes long.

VI. CONCLUSION

In this paper, we considered the quantum effects from the different moduli space in the scattering process. In three spatial dimensions, we obtained the phase shift and the deflection angle in the WKB approximation. Then we revealed the quantum effects in scattering process for $a^2 = 1/3$. We consider that these effects are due to the difference of the moduli space structure. Thus, the quantum effects from the moduli space structure are obtained in the scattering process. Moreover, there are the resonance states in the cases of $a^2 = 2/5, 9/20$ and $1/2$. These are the remarkable quantum effects, because there is no resonance state in the scattering process on the ordinary black holes with no dilaton coupling. The incoming particle falling into the ordinary black holes is simply absorbed or coalesce. In the scattering process in the moduli space for the specific case, however, there is the resonance state.

In further study, we will consider the behavior in the different dimensions and for the other values of dilaton coupling a. The moduli space structures are different in the other dimensions and at the other values of dilaton coupling a. In (4+1) dimensions for $1 < a^2 < 4$, the moduli space structure resembles the structure in (3+1) dimensions for $1/3 < a^2 < 1$ [6]. Then we expect that the similar quantum effects are also arisen. The resonance

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is the particular state in the black hole moduli space. It seems that the effects of this resonance states on the quantum mechanics and conformal symmetry discussed in [4] should be investigated. In this work, we considered the quantum effects in scattering process with the WKB approximation. We will discuss more precise quantum mechanical analysis in further study.

-A-

-B-
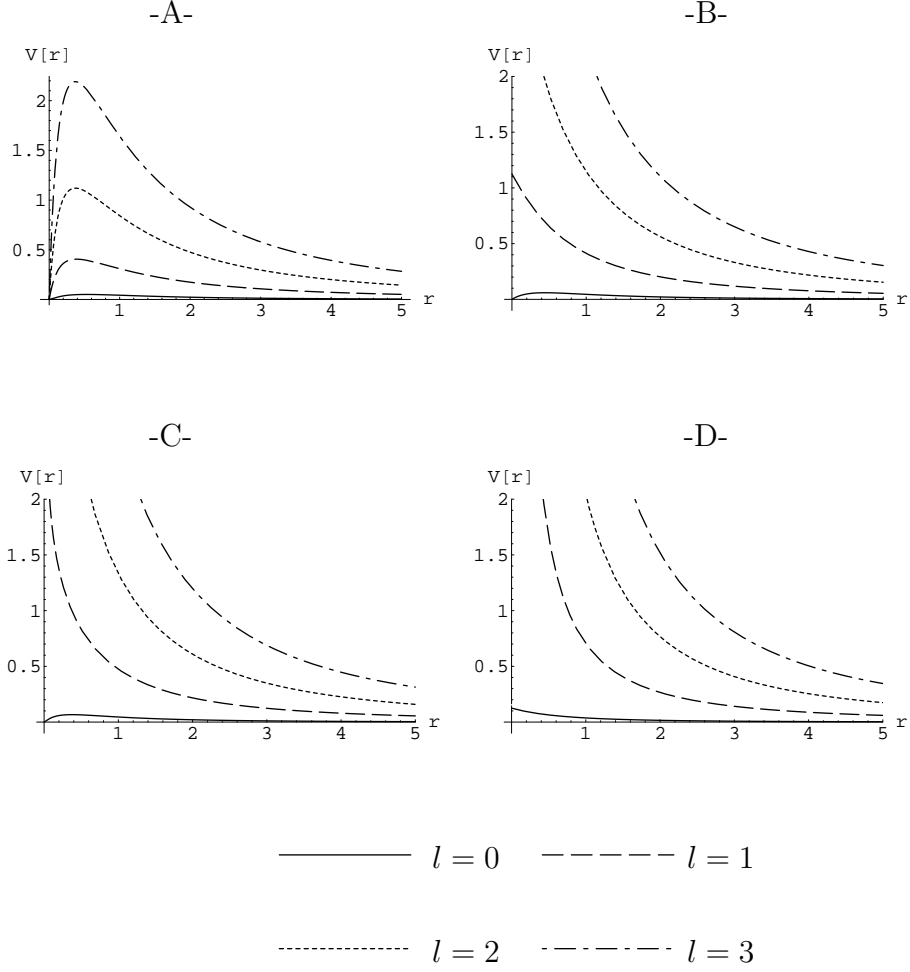
-C-

-D-

$l = 0$ ----- $l = 1$

············· $l = 2$ ·—·—·— $l = 3$

FIG. 1. The potential V as the function of r and angular momentum $l = 0, 1, 2, 3$, A for $a^2 = 0$, B for $a^2 = 1/3$, C for $a^2 = 1/2$ and D for $a^2 = 1$. We can find the potential barrier in the case of $a^2 = 0$ for all values of l and $a^2 = 1/3, 1/2$ for $l = 0$.
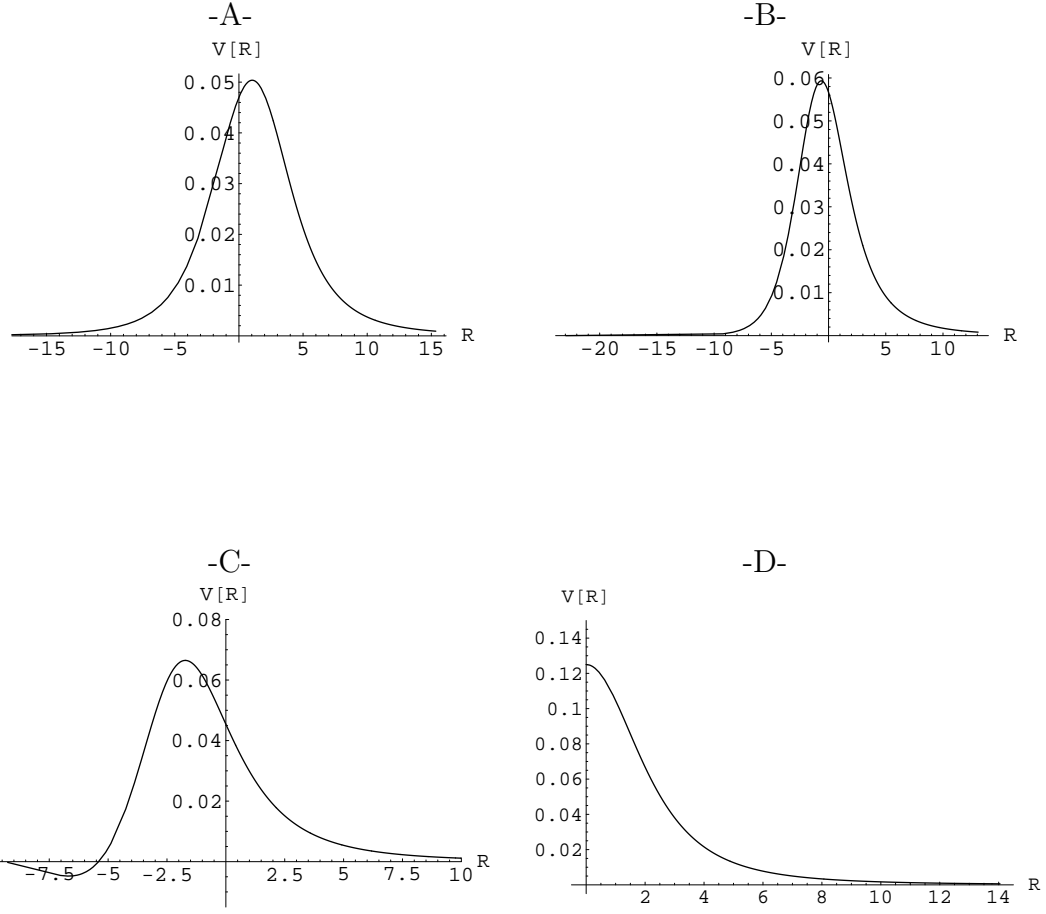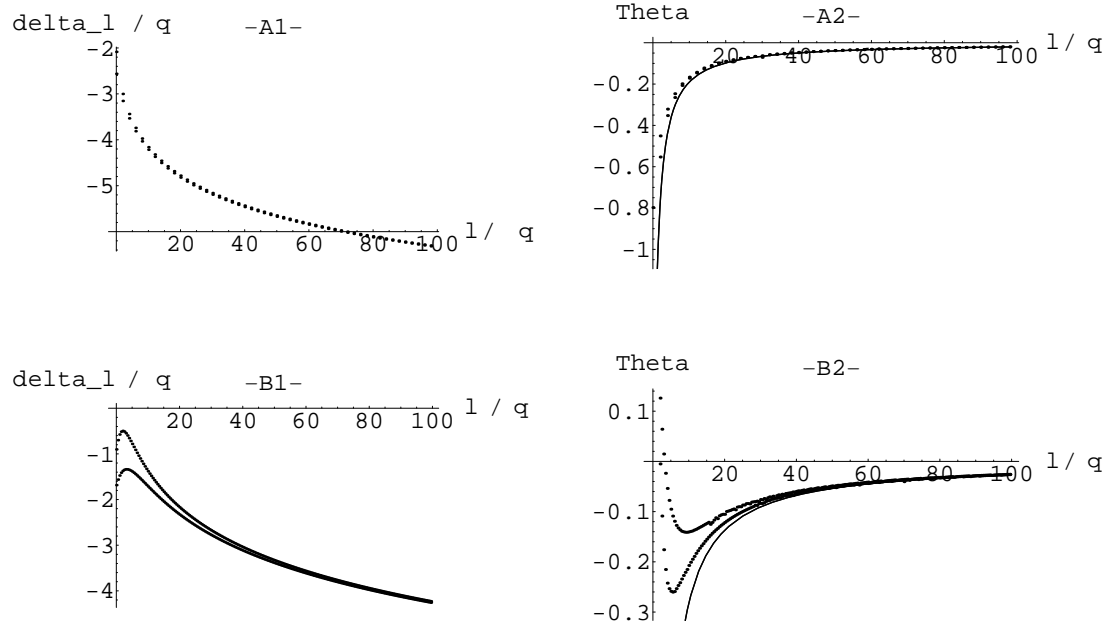
-A-

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FIG. 2. The potential V as the function of R when the angular momentum $l = 0$, A for $a^2 = 0$, B for $a^2 = 1/3$, C for $a^2 = 1/2$ and D for $a^2 = 1$.

FIG. 3. The phase shift and the deflection angle. A1 is a phase shift for $a^2 = 1$, A2 is a deflection angle for $a^2 = 1$, B1 and B2 are phase shift and deflection angle for $a^2 = 1/3$.
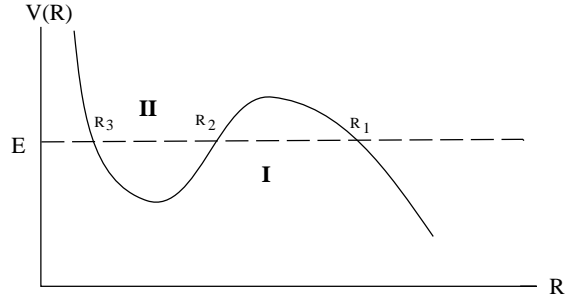

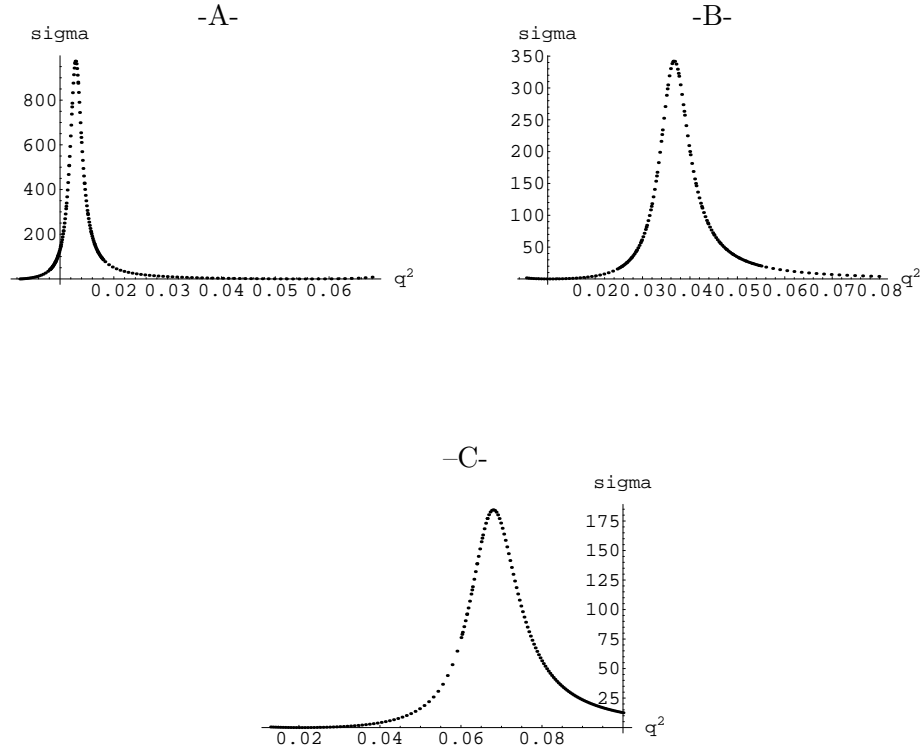FIG. 4. The form of typical Potential $V(R)$ with three turning points at $R = R_1, R_2, R_3$

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FIG. 5. The partial cross-section for $l = 0$, A for $a^2 = 2/5$, B for $a^2 = 9/20$ C for $a^2 = 1/2$.
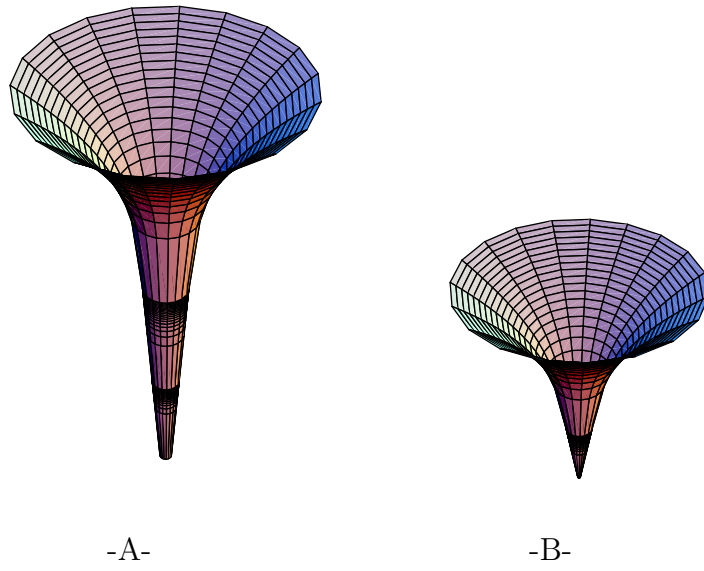

-A-   -B-

FIG. 6. The moduli space structure, A for $a^2 = 2/5$ and B for $a^2 = 1/2$.

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